Some exercises

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1. Show that the set $A = \{(x, y) \in \mathbb{R}^2 : x > y\}$ is open in $\mathbb{R}^2$.

> **Solution:** Let $f(x, y) = x - y$. f is a difference of continuous functions, then, it is continuous. The codomain of f is open (given that $x > y \Rightarrow f(x, y) = x - y > 0$) Finally, continuous functions take open sets to open sets, and therefore the domain must be open. Then A must be open.

2. Show that $B \subset U \subset \mathbb{R}^2$ is closed in U $\Leftrightarrow \forall \{x_n\}_{n\in\mathbb{N}} \to \bar{x}, \{x_n\}_{n\in\mathbb{N}} \subset B, \bar{x} \in B$

> **Solution:** ($\Rightarrow$)
>
> Let $\{x_n\}_{n\in\mathbb{N}} \in B, \{x_n\}_{n\in\mathbb{N}} \to \bar{x}$.
>
> Let $\bar{x} \in B^c$, given that B is closed, then by definition B^c is open. It follows that:
>
> $$\exists \epsilon > 0 \text{ tal que} B_\epsilon(\bar{x}) \subset B^c$$
>
> Using the definition of converging sequence, then $\exists n \in \mathbb{N}$ such that: $\parallel x_n - \bar{x} \parallel < \epsilon$
>
> Then, if we count the $n > N$ we obtain a convergent sequence with part of it in B^c, but by hypothesis, the sequence is contained in B, contradiction.
>
> ($\Leftarrow$)
>
> The sequence $\{x_n\}_{n\in\mathbb{N}} \subset B$ converges to $\bar{x} \in B$. Assume that B is not closed, then B^c is not open, and therefore $\exists \bar{x} \in B^c$ such that $\forall \epsilon > 0, B_\epsilon(\bar{x})$ has some elements of B.
>
> If you pick positive integers such that $\parallel x_n - \bar{x} \parallel < \frac{1}{n}$ with $x_n \in B$ then we have a convergent sequence with limit $\bar{x}$ and $\{x_n\}_{n\in\mathbb{N}} \in B$ with $\bar{x} \in B^c$, contradiction.

3. Given $U \subset \mathbb{R}^n$, if the function $g : U \to \mathbb{R}$ is continuous in U, then $\{x \in U : g(x) \geq 0\}$ is closed in U. Show that you cannot change the *then* with an *if and only if* (i.e. $\not\Leftarrow$).

> **Solution:** It is enough to show that $\{x \in U : g(x) \geq 0\}$ closed in U is not enough to show that $g : U \to \mathbb{R}$ is continuous in U. Even further, it is enough to show some g that satisfies $\{x \in U : g(x) \geq 0\}$ closed in U. Take $U = [-3, 3] \subset \mathbb{R}^n$ such that $g(x) = -1$ when $x \in [-3, 1)$ and $g(x) = 1$ when $x \in [1, 3]$. $\{x \in U : g(x) \geq 0\}$ is closed in U, $U \subset \mathbb{R}^n$ but g is not continuous.

4. Show: For $U \subset \mathbb{R}$ compact and f continuous, then $f : U \to \mathbb{R}$ is uniformly continuous.

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*Any mistake in the solutions is of the exclusive my responsibility.

Solution: Let $B(x)$ an open ball in U, centered in x with radius $\delta(x)/2$. U is the union of a set of these open balls. For every point in the set, you can create one of these balls. However, as U is compact we can choose a finite set of points that serve as center for these balls, such that:

$$U = B(x_1) \bigcup B(x_2) \bigcup B(x_3) ... \bigcup B(x_n)$$

If you choose *delta* as the lowest of the $\delta(x)/2$, we can be certain that we have satisfied the requirement (that the intersection is contained in the set). Now take p and $q \in U$, such that $d(p, q) < \delta$, for some x_i, you have that $p \in B(x_i)$. Then $d(p, x_i) < \delta(x_i)/2$ However,

$$d(p_i, q) \leq d(x_i, q) + d(p, q) < \delta(x_i)/s + \delta \leq \delta(x_i)$$

Then wen use $\delta = \min\{\delta(x_i)/2\}_{i \in \{1,...,n\}}$.

$$d(x_i, p) + d(x_i, q) < \delta(x_i)$$

Now using point continuity, given $\delta(x_i)$ we choose it, and we have $d(f(x_i), d(p)) < \frac{\epsilon}{2}$ y $d(f(x_i), f(q)) < \epsilon/2$.

$$d(f(p), f(q)) \leq d(f(x_i), f(p)) + d(f(x_i), f(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

5. Show that $\emptyset$ and $\mathbb{R}^n$ are convex.

Solution: $\emptyset \rightarrow$ You cannot show that is not convex, as you cannot pick two elements in the set such that do not satisfy convexity. $\mathbb{R}^n \rightarrow$ Sea a y b $\in \mathbb{R}^n, t \in [0, 1]$ $ta + (1 - t)b$, because of the axioms of $\mathbb{R}^n$ ta belongs to $\mathbb{R}^n$, and the summation as well, then it is convex (and a vector space!).

6. Given A and B convex, subsets of $\mathbb{R}^n$, show that:

$$A + tB := \{a + tb : a \in A, b \in B\}$$

is convex $\forall t \in \mathbb{R}$.

Solution: Let $a_i \in A$, $b_i \in B$ y $z_i = a_i + tb_i \in A + tB$ $i = 1, 2$. Let's show that $\{\lambda z_1 + (1 - \lambda) z_2\} \in A + tB$ $\lambda \in (0, 1)$ $\forall t \in \mathbb{R}$, $z_1, z_2 \in A + tB$

We know that

$$\lambda z_1 + (1 - \lambda) z_2 = \lambda a_1 + \lambda t b_1 + (1 - \lambda) a_2 + (1 - \lambda) t b_2 = \underbrace{\lambda a_1 + (1 - \lambda) a_2}_{\in A, convex} + t \underbrace{\lambda (b_1 + (1 - \lambda) b_2)}_{\in B, convex}$$

Then, it belongs to the set.

7. An arbitrary intersection of convex sets is convex.

Solution: Let a and b elements in the intersection of convex sets, then $(\lambda a + (1 - \lambda) b)$ belongs to each set, as these are convex. If it belongs to all, then it also belongs to the intersection. Then, the intersection is convex.

8. Show that a vector space is convex.

Solution: By definition, a vector space contains elements that can be added up and multiplied by a scalar. In particular, convex combinations of its elements belong to the same vector space.

Let a and b belong to a vector space, λa and $(1-\lambda)b$ are well defined and belong to the vector space as $\lambda \in \mathbb{R}$, and particularly $\lambda \in (0,1)$.

As the summation is closed in the vector space then the summation $\lambda a + (1-\lambda)b$ belongs to the vector space, concluding the proof.

9. $C \subset \mathbb{R}^n$ convex $\Rightarrow$ The closure is convex.

Solution: If C is closed, then it is trivial. Let C be not closed.

Let the closure to be not convex, while C is convex.

Let $a, b \in \bar{C} \backslash C$ $\lambda \in (0,1)$, $\lambda a + (1-\lambda)b \notin \bar{C}$ Then $\exists \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \subset C$ such that $\{\lambda a_n + (1-\lambda)b_n\} \in C \forall n$ A linear combination is continuous, so if I get close to a and b, the linear combination of these elements in the sequence must go through the closure, but C is convex, then contradiction.

10. There is a market with J assets traded by subjects that maximize their wealth. Each subject can create a portfolio $\theta = (\theta_1, ..., \theta_J) \in \mathbb{R}^J$, where $\theta_j > 0$ means that the subject bought θ_j units of asset j, while $\theta_j < 0$ means that the subject promised in the future to pay the market value of θ_j units of asset J (or, *shorted* θ_j units of the asset j). The value of each unit of asset j is given by $q_j \geq 0$. Assume there is uncertainty regarding the future price of the assets. Consider that there are S different possible states of nature. On state $s \in \{1, ..., S\}$ the asset $j \in \{1, ..., J\}$ has a value of $\mathbb{R}_{s,j}$ per unit. In sum, a portfolio $\theta \in \mathbb{R}^J$ will be worth, for a specific state of nature $s \in \{1, ..., S\}$ $\sum_{j=1}^{J} R_{s,j}\theta_j$.

As said, the subjects try to maximize their wealth, it is expectable that there are no positions that could generate unlimited wealth without taking some risks. Formally, let's say that *there is no arbitrage* if there is no portfolio $\theta \in \mathbb{R}^J$ such that $\sum_{j=1}^{J} q_k\theta_j \leq 0$ and, for each state of nature $s \in \{1, ..., S\}$, $\sum_{j=1}^{J} R_{s,j}\theta_j \geq 0$, with at least one of the inequalities being strict. Put in words, there is no way of (i) get more money today without sacrificing (expected) future wealth; or (ii) without paying nothing today increasing future wealth.

The following steps are suggested:

(a) Let

$$A = \begin{pmatrix} -q_1 & \dots & -q_J \\ R_{1,1} & \dots & R_{1,J} \\ \vdots & & \vdots \\ R_{S,1} & \dots & R_{S,1} \end{pmatrix}$$

Show that, without aribtrage, the set $C := \{z \in \mathbb{R}^{S+1} : \exists \theta \in \mathbb{R}^J, z = A\theta\}$ is disjoint with $C_\epsilon := \{z \in \mathbb{R}_+^{S+1} :\| z \| \in [\epsilon, 2]\}$, $\epsilon > 0$.

Solution: $C :=$

$$z = \begin{pmatrix} \sum_{i=1}^{J} -q_i\theta_i \\ \sum_{i=1}^{J} R_{1i}\theta_i \\ \vdots \\ \sum_{i=1}^{J} R_{si}\theta_i \end{pmatrix}$$

$$\rightsquigarrow \quad z_1 = -q_1\theta_1 - q_2\theta_2 - ... - q_J\theta_J$$
$$\vdots$$
$$z_i = R_{i1}\theta_1 + R_{i2}\theta_2 - ... + R_{iJ}\theta_J$$
$$\vdots$$
$$z_s = R_{s1}\theta_1 + R_{s2}\theta_2 - ... + R_{sJ}\theta_J$$

If there is no arbitrage then:

$$\nexists \theta \in \mathbb{R}^J \text{ such that } \sum_{j=1}^{J} q_j\theta_j \leq 0 \text{ and } s \in \{1...S\} \sum_{j=1}^{J} R_{sj}\theta j \geq 0$$

If you could find some θ such that happens, then $z \in \mathbb{R}_+^{s+1}$. As it doesn't happen, it can be zero, or with some negative coordinate, then $z \in \mathbb{R}^{s+1} \backslash \mathbb{R}_+^{s+1}$. It follows that $C_\epsilon \cap C = \emptyset$ as $C_\epsilon \subset \mathbb{R}_+^{s+1}$ and therefore they are disjoint.

(b) Show that C and C_ϵ are convex, non empty and closed. Show that C_ϵ is compact.

Solution:
The empty set case is trivial because $0 \in C$ and given $z \in C_\epsilon$, in particular a vector such that $\{c \in \mathbb{R}_+^{s+1} : 0 \leq \| c \| \leq 2\} \Rightarrow \sqrt{\sum_1^{s+1}(c_i)^2} \leq 2$

$$\sum_{i=1}^{s+1}(c_i)^2 \leq 4$$

For example $c_j = 2$ and all the others 0 satisfies the requirement.
Convexity,
The set C_ϵ is a closed ball in $\mathbb{R}_+^{S+1}$, that is a convex set. Then C_ϵ is convex.
C: Let a and $b \in C$, $\lambda \in [0,1]$

$$\begin{aligned}\lambda a + (1-\lambda)b &= \lambda A\theta_c + (1-\lambda)A\theta_b \\ &= A\theta_b + \lambda(A\theta_c - A\theta_b) = A[\theta_b + \lambda(\theta_c - \theta_b)] \\ &= A\underbrace{(\theta_b(1-\lambda) + \lambda\theta_c)}_{\in \mathbb{R}^J} \Rightarrow \exists \theta \in \mathbb{R}^J \text{ such that the set is convex}\end{aligned}$$

Closedness,
C_ϵ is closed in $\mathbb{R}_+^{s+1}$. It is enough to show that $\mathbb{R}_+^{s+1}$ is open if we exclude C_ϵ, that is, to show that $\{z \in \mathbb{R}_+^{s+1} : \| z \| < 2\}$, but the function $\| \cdot \|$ is continuous, and its codomain in the set is $(2, +\infty)$, then the set is open. By definition of a closed set, C_ϵ is closed.
C is closed, because z is a linear transformation of a vector space. The linear transformation of a vector space is itself a vector space, and a vector space is always closed.
That C_ϵ is compact is trivial, as the norm of all of its element is bounded by 2. Then, as we have shown, C_ϵ is closed and bounded, then it is compact.

(c) Show that there is $p >> 0$ such that $pz \leq 0$, for each $z \in C$. [Hint: Lookout the **hyperplane separation theorem**]

Solution:
From the previous result, we can apply the hyperplane separation theorem. In particular we have

$$pa < c < pb \quad \forall (a, b) \in C \times \ C_\epsilon, \text{ fore some } c \in \mathbb{R}$$

Where we can chose p.
Of curse we cannot set $p = 0$ as $0 < 0$ is false. The elements in C_ϵ have at least one positive coordinate. Then, we can bound it from below with 0. However 0 is feasible in C, and we know that $pz \leq 0$. As z has some negative coordinate, the it could be that if p would have negative coordinates, we could obtain something "greater or equal" than 0. Finally, as p cannot be 0, then $p >> 0$.

(d) Show that $pz = 0$ for each $z \in C$ (remember that C is a vector subspace).

Solution: Let's assume that $pz < 0$. If that is true, then $p(-z) > 0$, but as C is a vector subspace, $-z$ must belong to the vector subspace, but $pz < 0 \, \forall z \in C$. Contradiction. The only possibility is that $pz = 0$.

(e) Conclude the proof.

Solution: Given that $pz = 0$ and that $p >> 0$ then:

$$pz = 0$$

$$-p_1 \sum_{j=1}^{J} q_j \theta_j + p_2 \sum_{j=1}^{J} R_{1,j} \theta_j + \ldots + p_{s+1} \sum_{j=1}^{J} R_{S,j} \theta_j = 0$$

$$p_2 \sum_{j=1}^{J} R_{1,j} \theta_j + \ldots + p_s + 1 \sum_{j=1}^{J} R_{S,j} \theta_j = p_1 \sum_{j=1}^{J} q_j \theta_j$$

$$\frac{p_2}{p_1} \sum_{j=1}^{J} R_{1,j} \theta_j + \ldots + \frac{p_{s+1}}{p_1} \sum_{j=1}^{J} R_{S,j} \theta_j = \sum_{j=1}^{J} q_j \theta_j$$

If we redefine, $\frac{p_j}{p_1} = \gamma_{j-1}$ then we have:

$$\gamma_1 \sum_{j=1}^{J} R_{1,j} \theta_j + \ldots + \gamma_S \sum_{j=1}^{J} R_{S,j} \theta_j = \sum_{j=1}^{J} q_j \theta_j$$

The this is equivalent to:

$$\gamma^T R \theta = q^T \theta$$
$$\gamma^T R = q^T$$

The for each element i we have that:

$$\sum_{j=1}^{J} \gamma_i R_{i,j} = q_i$$

For the next questions, consider that a function is said to be **quasiconcave** if $f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$

11. Show that any function $f : \mathbb{R}^n \to \mathbb{R}$ as $f(x) = ax + b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, is quasiconcave.

Solution:

$$
\begin{aligned}
f(\lambda x_1 + (1-\lambda)x_2) &= a\lambda(x_1 - x_2) + f(x_2) + \lambda b - \lambda b \\
&= \lambda f(x_1) - \lambda f(x_2) + f(x_2) \\
&= \lambda f(x_1) + (1-\lambda) f(x_2) \\
\Rightarrow f(\lambda x_1 + (1-\lambda)x_2) &= \lambda f(x_1) + (1-\lambda) f(x_2) \\
\text{si, } f(x_1) &> f(x_2) \\
\geq f(x_2) = \min f(x_1), f(x_2)& \\
\text{si, } f(x_1) &< f(x_2) \\
\geq f(x_1) = \min f(x_1), f(x_2)& \\
\Rightarrow f(\lambda x_1 + (1-\lambda)x_2) &\geq \min f(x_1), f(x_2)
\end{aligned}
$$

12. A monotone function (increasing or decreasing) is always quasiconcave.

Solution: Having $x_1 \geq x_2$ is clear that $\lambda x_1 + (1-\lambda)x_2 \geq x_2$, Then:

$$
\begin{aligned}
f(\lambda x_1 + (1-\lambda)x_2) \geq f(x_2), &\text{ if } f \text{ is increasing.} \\
\Rightarrow f(x_2) &= \min\{f(x_1), f(x_2)\} \\
f(\lambda x_1 + (1-\lambda)x_2) \geq f(x_1), &\text{ if } f \text{ is decreasing.} \\
\Rightarrow f(x_2) &= \min\{f(x_1), f(x_2)\}
\end{aligned}
$$

In both cases $f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\}$, then f is quasiconcave.

13. Any concave function is quasiconcave.

Solution: If:

$$f(x_1) < f(x_2) \Rightarrow \lambda f(x_1) + (1-\lambda)f(x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_1) = f(x_1) = \min\{f(x_1), f(x_2)\}$$

The proof in the opposite case follows in the same way.

14. Given function $f : U \subset \mathbb{R}^n \to \mathbb{R}$, where U is convex, f is quasiconcave in U if and only if, for each a in $\mathbb{R}$ the set $U_a = \{x \in U : f(x) > a\}$ is convex.

Solution:

($\Rightarrow$) hypothesis:

f is quasiconcave in $U, a \in \mathbb{R}$, for when $U = \emptyset$, is trivial. Let $U \neq \emptyset$.

Let $x_1, x_2 \in U$ $a \in \mathbb{R}$, $\lambda \in (0,1)$.

$$z_\lambda = \lambda x_1 + (1-\lambda)x_2 \quad \in U \text{ , because } U \text{ is convex}$$

using f's quasiconcavity

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\} \to \quad \text{(easy to see by the def. of the set)}$$
$$\min\{f(x_1), f(x_2)\} > a \quad \text{then}$$
$$f(\lambda x_1 + (1-\lambda)x_2) > a \Rightarrow f(\lambda x_1 + (1-\lambda)x_2) \in U_a$$
$$\Rightarrow U_a \text{ is convex}$$

($\Leftarrow$) Now the hypothesis is: U_a is convex for each $a \in \mathbb{R}$. Let $x_1, x_2 \in U$, $\{x_1, x_2\} \subset U_{\min\{f(x_1), f(x_2)\}}$ then for each $\lambda \in (0,1)$

$$\lambda x_1 + (1-\lambda)x_2 \quad \in \quad \min\{f(x_1), f(x_2)\}$$
$$\Rightarrow f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\} \quad \forall \lambda \in (0,1)$$

Concluding the proof.

15. Given $(\alpha, \beta) >> 0$, $(x, y) \in \mathbb{R}_+$, the function $f(x,y) = x^\alpha y^\beta$ is strictly quasiconcave.

Solution: Using the result in exercise (14), if $a = 0$, then it is easy to see that f is quasiconcave. ($\mathbb{R}^2_+$ is convex).

16. Given $a \in \mathbb{R}^n$, $f(x) = -\parallel x - a \parallel$ is strictly quasiconcave.

Solution: Let $x, y \in \mathbb{R}^n$

$$\begin{aligned} z &= \lambda x + (1-\lambda)y \\ g(x) &= -f(x) \\ g(z) &= \parallel \lambda x + (1-\lambda)y - a \parallel \\ &= \parallel \lambda(x-a) + (1-\lambda)(y-a) \parallel \\ &\leq \lambda \parallel x - a \parallel + (1-\lambda) \parallel y - a \parallel \end{aligned}$$

Assume an x further from a than from y.

$$\begin{aligned} g(z) &< g(x) \\ g(\lambda x + (1-\lambda)y) &< g(x) \\ \parallel \lambda x + (1-\lambda)y - a \parallel &< \parallel x - a \parallel \quad /-1 \\ -\parallel \lambda x + (1-\lambda)y - a \parallel &> -\parallel x - a \parallel \\ f(z) &> f(x) \end{aligned}$$

Because we assumed x further from a than from y,

$$\begin{aligned} g(x) &= \max\{g(x), g(y)\} \\ \Rightarrow f(x) &= \min\{f(x), f(y)\} \end{aligned}$$

Then $f(z) > \min\{f(x), f(y)\}$, concluding the proof.

1 Bonus Track, Proof fo the Caratheodory's Theorem

Definition 1.1. The covenxhull of a set S is the set that contains all the convex combinations of a finite number of elements of S.

If some $x \in \mathbb{R}^d$ is in the convexhull of P ($co(P)$), then, there is a subset $P' \subset P$ that has $d + 1$ or less elements such that x is in the $co(P)$.

Solution:

Let $x \in co(P)$. Then x is a convex combination of a finite number of elements of P:

$$x = \sum_{j=1}^{k} \lambda_j x_j$$

where each $x_j \in P$, each $\lambda_j \geq 0$ and it holds that

$$\sum_{j=1}^{k} \lambda_j = 1$$

Assume that $k > d+1$ (if not, there is nothing to prove). Then, the elements $(x_2 - x_1), ..., (x_k - x_1)$ are linearly dependent, that is there are scalars $\mu_2, ..., \mu_k$ such that:

$$\sum_{j=2}^{k} \mu_j (x_j - x_1) = 0$$

$$\sum_{j=2}^{k} \mu_j x_j - \sum_{j=2}^{k} \mu_j x_1 = 0$$

$$\sum_{j=2}^{k} \mu_j x_j = \sum_{j=2}^{k} \mu_j x_1$$

If we define

$$\mu_1 := -\sum_{j=2}^{k} \mu_j$$

$$\sum_{j=2}^{k} \mu_j x_j = \sum_{j=2}^{k} \mu_j x_1 \quad / + \mu_1 x_1$$

$$\sum_{j=1}^{k} \mu_j x_j = x_1 \sum_{j=1}^{k} \mu_j$$

And noting that:

$$x_1 \sum_{j=1}^{k} \mu_j = \mu_1 x_1 + \sum_{j=2}^{k} \mu_j x_1 = -x_1 \sum_{j=2}^{k} \mu_j + x_1 \sum_{j=2}^{k} \mu_j = x_1 \left[-\sum_{j=2}^{k} \mu_j + \sum_{j=2}^{k} \mu_j \right] = 0$$

Then

$$\sum_{j=1}^{k} \mu_j x_j = x_1 \sum_{j=1}^{k} \mu_j = 0$$

$$\sum_{j=1}^{k} \mu_j x_j = 0$$

$$\sum_{j=1}^{k} \mu_j = 0$$

And not all the μ_j are zero. Then, at least some $\mu_j > 0$. It follows that,

$$x = \sum_{j=1}^{k} \lambda_j x_j - \alpha \sum_{j=1}^{k} \mu_j x_j = \sum_{j=1}^{k} (\lambda_j - \alpha \mu_j) x_j$$

for some $\alpha \in \mathbb{R}$. In particular, the inequality will hold if α is defined as,

$$\alpha := \min_{1 \leq j \leq k} \left\{ \frac{\lambda_j}{\mu_j} : \mu_j > 0 \right\} = \frac{\lambda_i}{\mu_i}$$

Note that $\alpha > 0$ and for j between 1 and k,

$$\lambda_j - \alpha \mu_j \geq 0$$

in particular $\lambda_j - \alpha \mu_j = 0$, by definition of α. Then

$$x = \sum_{j=1}^{k} (\lambda_j - \alpha \mu_j) x_j$$

where each $(\lambda_j - \alpha \mu_j)$ is non negative, adds ip to 1 and even further, $\lambda_i - \alpha \mu_i = 0$. In other words x is a convex combination of at the most $k - 1$ elements of P.

This process can be repeated until x is represented as a convex combination of at the most $d + 1$ elements of P.